VEKTORANALYS

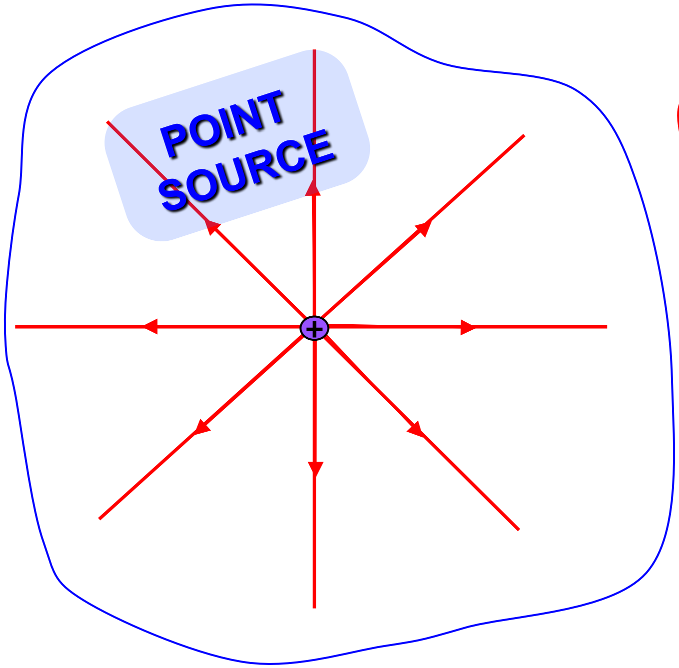
Kursvecka 6

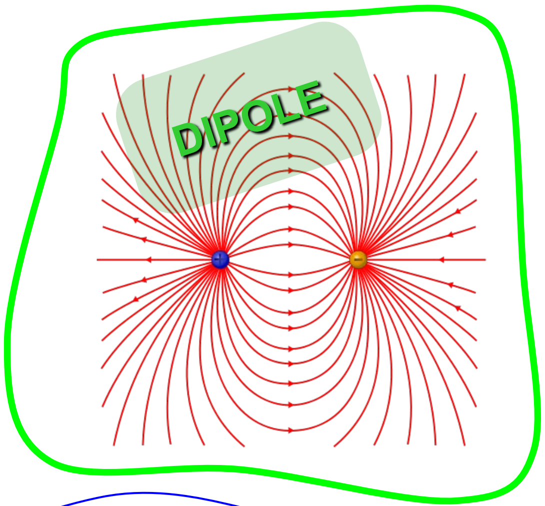
SOME SPECIAL VECTOR FIELDS
AND
LAPLACE AND POISSON EQUATIONS

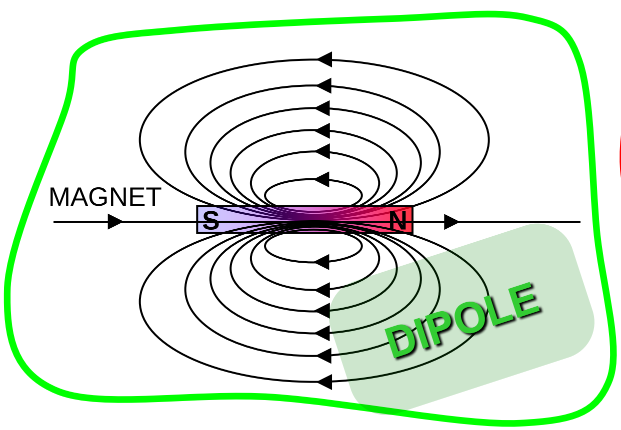
Kapitel 11-12
Sidor 123-150

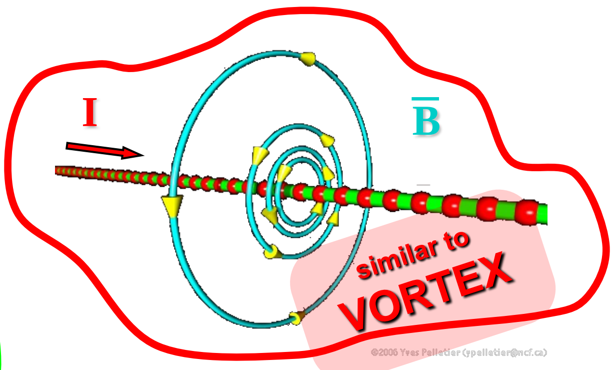
TARGET PROBLEM

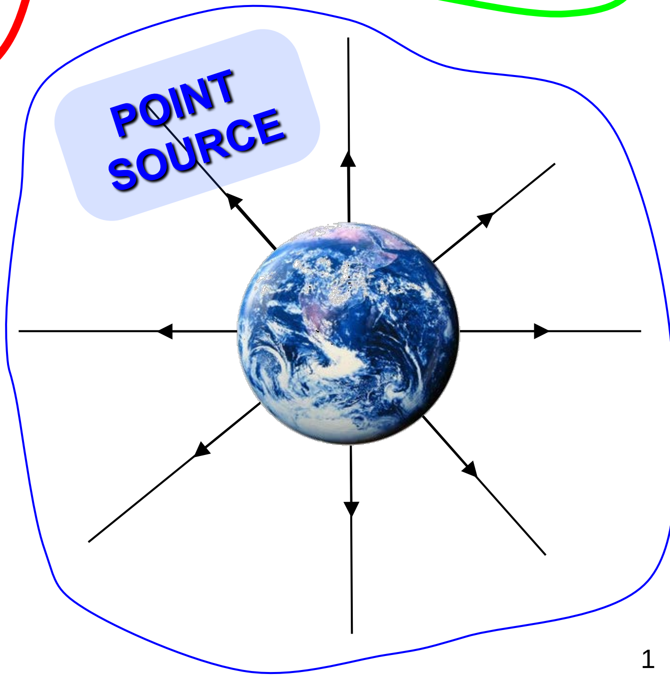
Some example of vector field sources in nature











TARGET PROBLEM

- **Point source** *(punktkällan)*

  It is a single identifiable localized source with negligible extent.

  In some particular conditions,
  (for example: 3D space, emission homogenous in all directions, no absorption and no loss…)
  the field produced by a point source decreases with r^{2}

- **Dipole source** *(dipolskällan)*

  Two identical but opposite sources (i.e. a source and a sink)
  separated by a distance d.

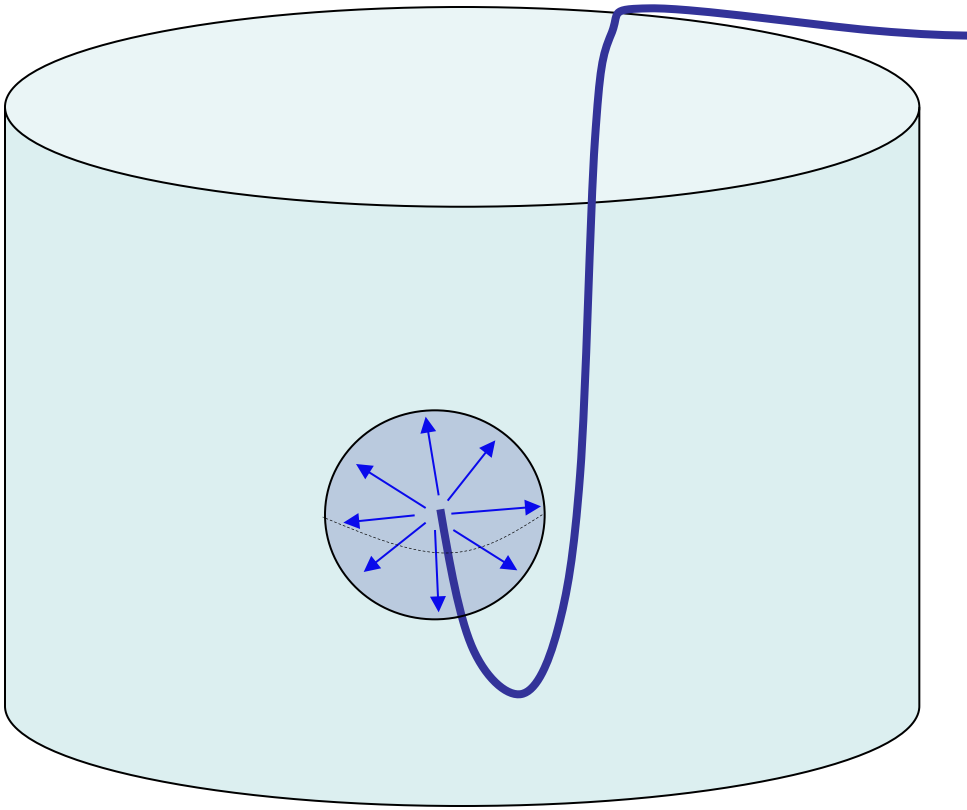
- **Vortex** *(virveltråden)*

  The velocity field in a water vortex
  Magnetic field around a straight wire
  …

POINT SOURCE

A single identifiable localized source with negligible extent.

Let’s calculate the velocity field of the water that flows from a thin pipe into a large pool.



Assumptions:

1- The source is homogeneous in time

i.e. the flow of the water from the pipe is constant: F=Volume/time=constant

2- The emission is homogeneous in all directions

3- No absorption, no losses

Then:

$$\left.\begin{array}{l} F = \bar{S} \cdot \bar{v} \\ \bar{S} = 4\pi r^2 \hat{e}_r \end{array}\right\} \Rightarrow \bar{v} = \frac{F}{4\pi r^2}\hat{e}_r$$

In a 3D space, the **vector field generated by a point source** is:

$$\boxed{\bar{A}(\bar{r}) = \frac{q}{r^2}\hat{e}_r}$$

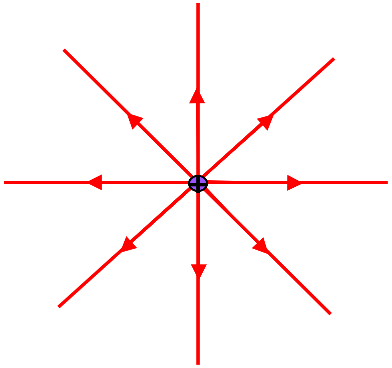
POINT SOURCE

The vector field generated by a point source located in the origin is:

$$\overline{A}(\overline{r}) = \frac{q}{r^2}\hat{e}_r$$

When the source is not in the origin:

$$\overline{A}(\overline{r}) = q\frac{\overline{r}-\overline{r}\,'}{\left|\overline{r}-\overline{r}\,'\right|^3} \qquad \textit{where } \overline{r}\,' \textit{ is the position of the source}$$

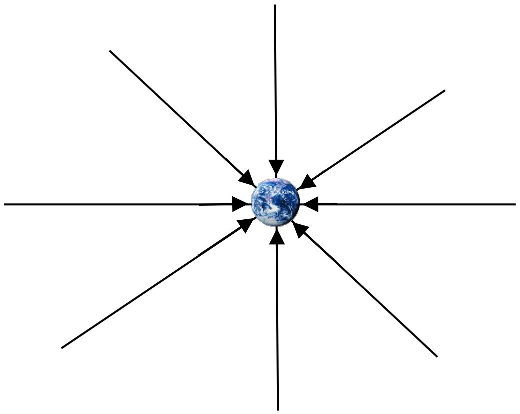


- Electrostatic field produced by a point charge:

$$\overline{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r \qquad \textit{with} \quad s = \frac{q}{4\pi\varepsilon_0}$$

- Gravitational field produced by a mass M:

$$\overline{g} = -GM\frac{1}{r^2}\hat{e}_r \qquad \textit{with} \quad s = -GM$$



POINT SOURCE

THEOREM 1 *(11.1 in the book)*

The flux produced by a point source through a closed surface S (with S boundary of the volume V) is:

$$\oiint_S \frac{q}{r^2}\hat{e}_r \cdot d\overline{S} = \begin{cases} 0 & \textit{If the source is outside } V \\ 4\pi q & \textit{If the source is inside } V \end{cases}$$

PROOF

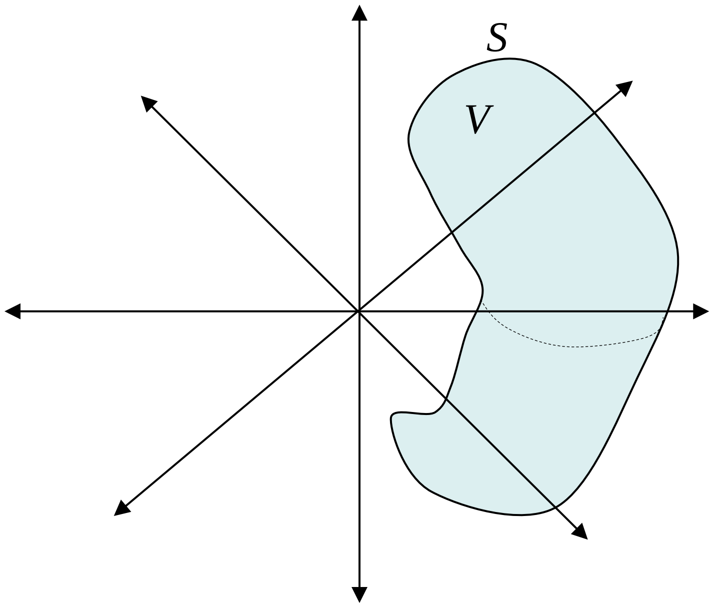
1. The origin is outside V

In V the field is continuously differentiable, so we can apply the Gauss' theorem:

$$\oiint_S \frac{q}{r^2}\hat{e}_r \cdot d\overline{S} = \iiint_V div\left(\frac{q}{r^2}\hat{e}_r\right)dV$$

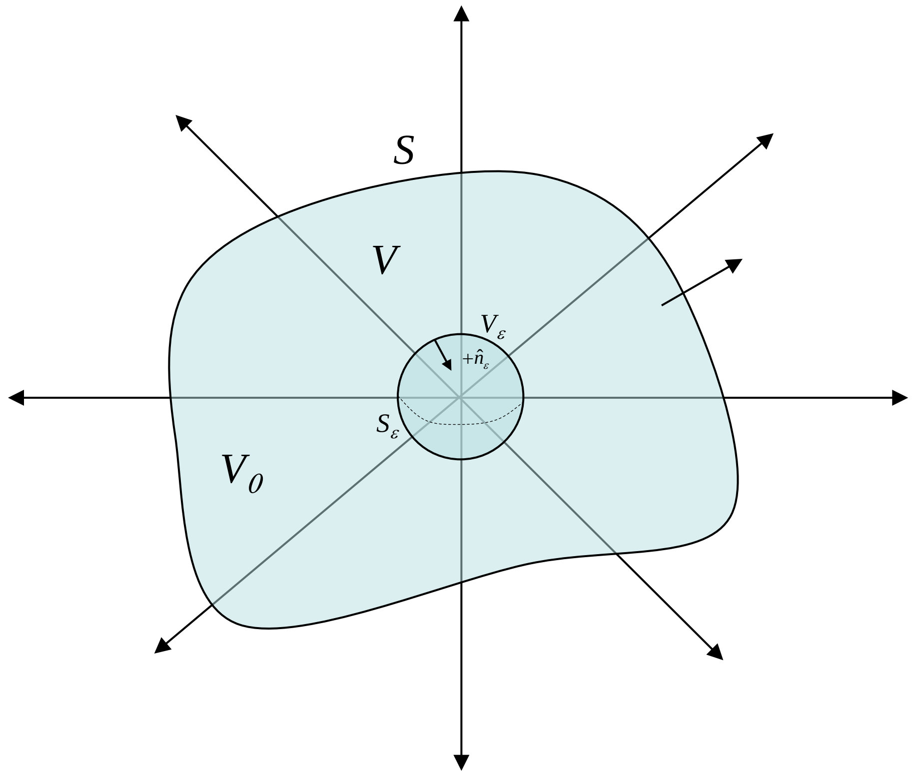
$$div\left(\frac{q}{r^2}\hat{e}_r\right) = \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{q}{r^2}\right) \qquad = 0$$

$$\Rightarrow \oiint_S \frac{q}{r^2}\hat{e}_r \cdot d\overline{S} = 0$$



2. The origin is inside V

The field is not continuous in V,
since the origin is a singular point.
So the Gauss' theorem cannot be applied.



But we can divide V into two volumes:

$$V = V_0 + V_\varepsilon$$

V_ε is a "small" sphere with radius ε
with centre on the source (the origin).
V_0 is the remaining part of V

$$\iint_S \frac{q}{r^2}\hat{e}_r \cdot d\bar{S} = \iint_{S+S_\varepsilon-S_\varepsilon} \frac{q}{r^2}\hat{e}_r \cdot d\bar{S} =$$

$$\iint_{S+S_\varepsilon} \frac{q}{r^2}\hat{e}_r \cdot d\bar{S} + \iint_{-S_\varepsilon} \frac{q}{r^2}\hat{e}_r \cdot d\bar{S} =$$

Gauss'theorem: V_0 does not contain the origin

$$\iiint_{V_0} \underbrace{div\left(\frac{q}{r^2}\hat{e}_r\right)}_{=0} dV - \iint_{S_\varepsilon} \frac{q}{r^2}\hat{e}_r \cdot d\bar{S} = -\iint_{S_\varepsilon} \frac{q}{r^2}\hat{e}_r \cdot \underbrace{(-\hat{e}_r)}_{\hat{n}_\varepsilon = -\hat{e}_r} dS = \iint_{S_\varepsilon} \frac{q}{\varepsilon^2} dS = \frac{q}{\varepsilon^2}\iint_{S_\varepsilon} dS = \frac{q}{\varepsilon^2}\underbrace{4\pi\varepsilon^2}_{\text{Area of the sphere with radius } \varepsilon} = 4\pi q$$

THE POTENTIAL OF A POINT SOURCE

The potential from a point source is: $\boxed{\phi = -\frac{q}{r} + const.}$

In fact: $grad\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\underbrace{\frac{\partial \phi}{\partial \theta}}_{=0}\hat{e}_\theta + \frac{1}{r\sin\theta}\underbrace{\frac{\partial \phi}{\partial \varphi}}_{=0}\hat{e}_\varphi = -q\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{e}_r = \frac{q}{r^2}\hat{e}_r$

ELECTROSTATIC FIELD FROM A POINT SOURCE

The electrostatic field from a point source is $\bar{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{e}_r$

The electrostatic potential is defined as: $\bar{E} = -grad\,\phi_E$

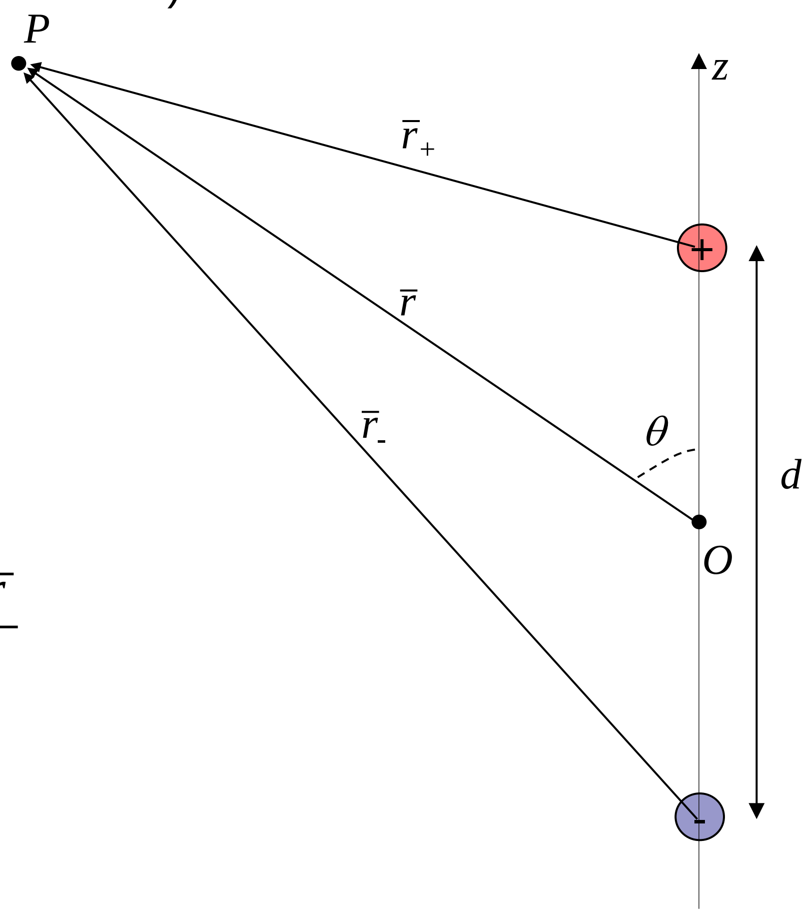
Therefore, the electrostatic potential is: $\boxed{\phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$

The flux of the electric field is: $\boxed{\iint_S \bar{E}\cdot d\bar{S} = \frac{q}{\varepsilon_0}}$ *where q is the total charge inside S*

DIPOLE SOURCE

Two identical but opposite sources (i.e. a source and a sink) separated by a distance d.

Assume that the origin is in the middle between the positive and the negative charge.



$$If \quad r >> d$$

$$r \approx r_+ \approx r_-$$

$$r_- - r_+ \approx d\cos\theta$$

The potential due to the dipole is:

$$\phi(\overline{r}) = \frac{q}{r_+} + \frac{-q}{r_-} = q\frac{r_- - r_+}{r_- r_+} \approx q\frac{d\cos\theta}{r^2} = q\frac{\overline{d}\cdot\overline{r}}{r^3}$$

Ideal dipole: $qd = constant$

The **dipole moment is defined as**: $\overline{p} \equiv q\overline{d}$

The field generated by the dipole is:

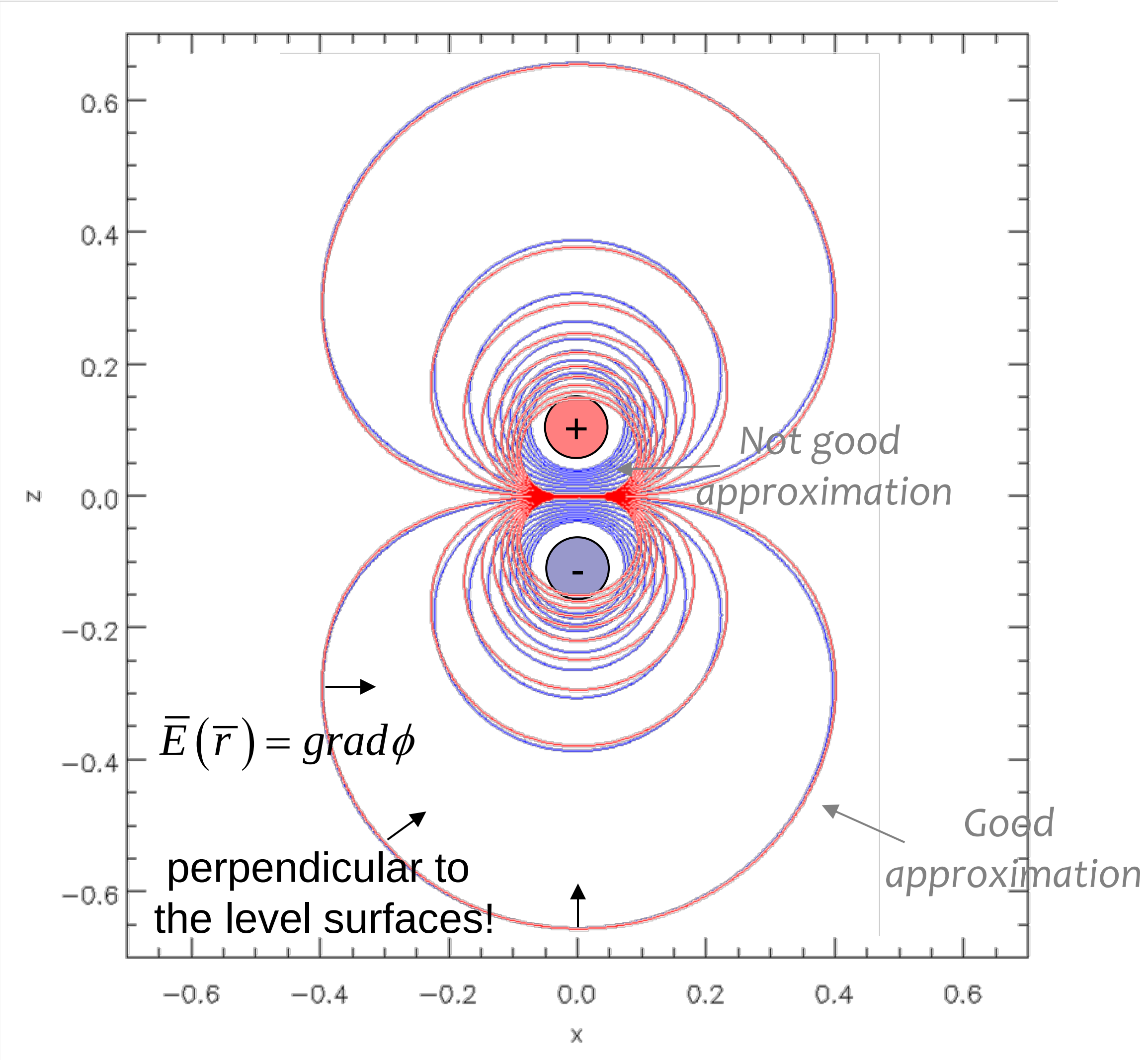
$$\overline{E}(\overline{r}) = -grad\phi = -grad\left(\frac{\overline{p}\cdot\overline{r}}{r^3}\right) = -\frac{\overline{p}}{r^3} + \frac{3(\overline{p}\cdot\overline{r})\overline{r}}{r^5}$$

$$\boxed{\begin{aligned}\phi(\overline{r}) &= \frac{\overline{p}\cdot\overline{r}}{r^3} \\ \overline{E}(\overline{r}) &= -\frac{\overline{p}}{r^3} + \frac{3(\overline{p}\cdot\overline{r})\overline{r}}{r^5}\end{aligned}}$$

DIPOLE SOURCE (example)

$$\phi(\bar{r}) = \frac{q}{r_+} - \frac{q}{r_-}$$

$$\phi(\bar{r}) = q\frac{d\cos\theta}{r^2}$$



VORTEX (or similar fields)

Example: The velocity field in a water vortex, the magnetic field around a straight wire…

The vector field generated by a vortex has the shape: $\bar{A}(\bar{r}) = \dfrac{k}{\rho}\hat{e}_{\varphi}$

THEOREM 2 *(11.2 in the book)*

The circulation of this vector field is

$$\oint_L \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} = 2\pi k N$$

where N is number of turns of L around the z-axis

N is positive if the turn is along +L

N is negative if the turn is along -L

PROOF

The field is singular on the *z-axis*.

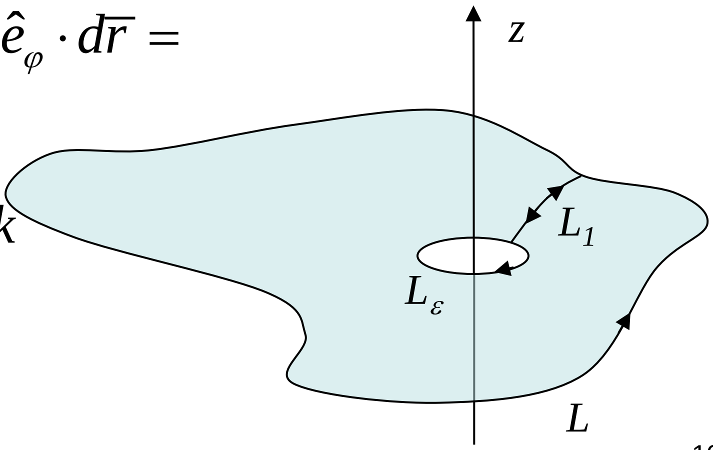
So the Stokes' theorem cannot be applied directly.

We consider a circular path L_{ε} with radius ε

$$\int_L \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} = \int_{L+L_{\varepsilon}-L_{\varepsilon}+L_1-L_1} \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} = \underbrace{\int_{L+L_{\varepsilon}+L_1-L_1}}_{} \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} + \int_{-L_{\varepsilon}} \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} =$$

$$\iint_S rot\left(\frac{k}{\rho}\hat{e}_{\varphi}\right) \cdot d\bar{S} + \int_{-L_{\varepsilon}} \frac{k}{\rho}\hat{e}_{\varphi} \cdot d\bar{r} = \int_0^{2\pi} \frac{k}{\varepsilon} \underbrace{\varepsilon\hat{e}_{\varphi} \cdot \hat{e}_{\varphi} d\varphi}_{d\bar{r} = -\varepsilon\hat{e}_{\varphi} d\varphi} = 2\pi k$$

Closed path that does not contain the z-axis.
We can apply the Stokes' theorem!



WHICH STATEMENT IS WRONG?

1- The vector field $\frac{q}{r^2}\hat{e}_r$ is produced by a point source (yellow)

2- The vector field $\frac{k}{\rho}\hat{e}_\varphi$ can represent the velocity field of a vortex (red)

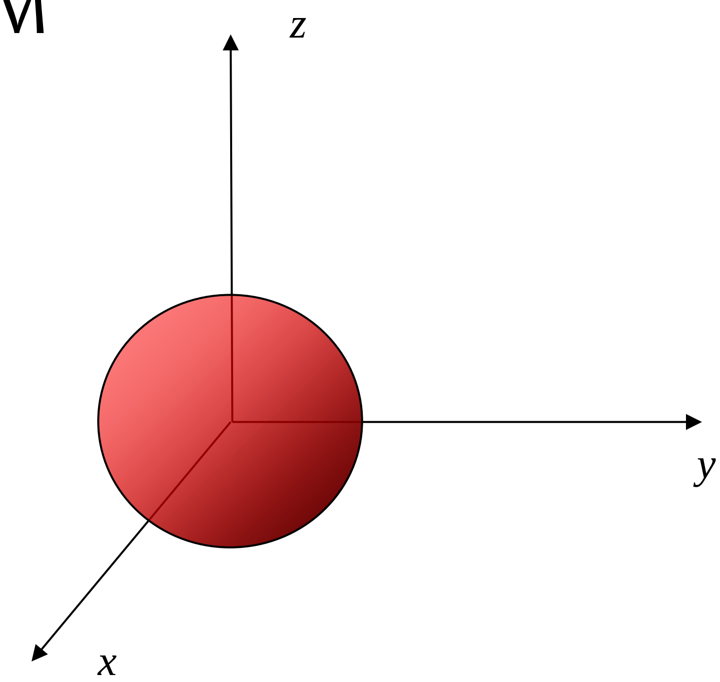
3- The flux of the field from a point source is always (green)

$$\iint_S \frac{s}{r^2}\hat{e}_r \cdot d\overline{S} = 4\pi s$$

4- The circulation $\int_L \frac{k}{\rho}\hat{e}_\varphi \cdot d\overline{r} = 2\pi k$ if L has only one turn around z (blue)

LAPLACE AND POISSON EQUATIONS

TARGET PROBLEM



A sphere has radius R and charge density $\rho=\rho_c$.
Calculate:
- the electric field and
- the electrostatic potential

inside and outside the sphere.

From the electromagnetic theory course:

$$\nabla \cdot \overline{E} = \frac{\rho_c}{\varepsilon_0}$$

$$\overline{E} = -\nabla \phi_E$$

Therefore: $$\nabla^2 \phi_E = -\frac{\rho_c}{\varepsilon_0}$$

This equation is an example of:

Laplace's equation $\boxed{\nabla^2 \phi = 0}$

Poisson's equation $\boxed{\nabla^2 \phi = \rho}$

SYMMETRIC SOLUTIONS

OF THE

LAPLACE EQUATION $\nabla^2\phi = 0$

PLANAR SYMMETRY $\phi = \phi(x)$ (NO y and z dependences)

In cartesian coord.

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\boxed{\frac{d^2\phi(x)}{dx^2} = 0} \Rightarrow \boxed{\phi(x) = ax + b}$$

CYLINDRICAL SYMMETRY $\phi = \phi(\rho)$ (NO φ and z dependences)

In cylindrical coord.

$$\nabla^2\phi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\boxed{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi(\rho)}{d\rho}\right) = 0} \Rightarrow \rho\frac{d\phi(\rho)}{d\rho} = a$$

$$\Rightarrow \boxed{\phi(\rho) = a\ln\rho + b}$$

SPHERICAL SYMMETRY $\phi = \phi(r)$ (NO θ and φ dependences)

In spherical coord.

$$\nabla^2\phi = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right)$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi(r)}{dr}\right) = 0} \Rightarrow r^2\frac{d\phi(r)}{dr} = a$$

$$\boxed{\Rightarrow \quad \phi(r) = -\frac{a}{r} + b}$$

LAPLACE AND POISSON EQUATIONS

THEOREM 1 *(12.2 in the book)*

> If ϕ has continuous second derivatives in the volume V and $\phi=0$ on the surface S that encloses V, then the solution to the Laplace equation $\nabla^2\phi=0$ is:
>
> $\phi(x,y,z)=0$ in V

PROOF

We know: $\nabla\cdot\left(f\,\overline{v}\right)=\left(\nabla f\right)\cdot\overline{v}+f\,\nabla\cdot\overline{v}$ **(ID2)**

$$\left.\begin{aligned} f&=\phi\\ \overline{v}&=\nabla\phi\end{aligned}\right\}\Rightarrow\ \nabla\cdot\left(\phi\nabla\phi\right)=\nabla\phi\cdot\nabla\phi+\phi\left(\nabla\cdot\nabla\phi\right)=\left(\nabla\phi\right)^2+\phi\underbrace{\nabla^2\phi}_{=0}$$

$$\Rightarrow\ \nabla\cdot\left(\phi\nabla\phi\right)-\left(\nabla\phi\right)^2=0\quad\Rightarrow\quad\iiint_V\left[\nabla\cdot\left(\phi\nabla\phi\right)-\left(\nabla\phi\right)^2\right]dV=0$$

Gauss' theorem ||

$$\underbrace{\iint_S\phi\nabla\phi\cdot d\overline{S}}_{\substack{=0\\ \text{because }\phi=0\text{ on }S}}-\iiint_V\underbrace{\left(\nabla\phi\right)^2}_{\geq 0}dV=0\quad\Rightarrow\quad\phi=0$$

DIRICHLET BOUNDARY CONDITIONS

$$\nabla^2\phi = \rho$$
$$\phi = \sigma \quad on\ S$$

Dirichlet boundary condition

What can we say about the solution?

THEOREM 2 *(12.3 in the book)*

The Poisson's equation $\nabla^2\phi = \rho$ in the volume V with boundary condition $\phi = \sigma$ on the surface S that encloses V has only one solution.

PROOF Let's assume that ϕ_1 and ϕ_2 are two solution:

$$\nabla^2\phi_1 = \rho \quad and \quad \phi_1 = \sigma \quad on\ S$$
$$\nabla^2\phi_2 = \rho \quad and \quad \phi_2 = \sigma \quad on\ S$$

Let's now define $\phi_0 = \phi_1 - \phi_2$

$$\left.\begin{aligned}\nabla^2\phi_0 &= \nabla^2(\phi_1 - \phi_2) = \overbrace{\nabla^2\phi_1}^{\rho} - \overbrace{\nabla^2\phi_2}^{\rho} = 0 \\ \phi_0 &= \underbrace{\phi_1}_{\sigma} - \underbrace{\phi_2}_{\sigma} = 0 \quad on\ S\end{aligned}\right\}$$

Due to theorem 1: $\underbrace{\phi_0 = 0 \ in\ V}$

$$\Downarrow$$

$\phi_1 = \phi_2$ in V

NEUMANN BOUNDARY CONDITIONS

$$\nabla^2\phi=\rho$$
$$\frac{\partial\phi}{\partial n}=\hat{n}\cdot\nabla\phi=\gamma \quad on\ S$$

Neumann boundary condition

What can we say about the solution?

THEOREM 3 *(12.4 in the book)*

The solution to the Poisson's equation $\nabla^2\phi=\rho$ in V with boundary condition $\hat{n}\cdot\nabla\phi=\gamma$ on S is not unique. If ϕ_s is a solution then ϕ_s+c is also solution where c is an arbitrary constant.

PROOF Let's assume that ϕ_1 and ϕ_2 are two solution:

$$\nabla^2\phi_1=\rho \quad and \quad \hat{n}\cdot\nabla\phi_1=\gamma \quad on\ S$$
$$\nabla^2\phi_2=\rho \quad and \quad \hat{n}\cdot\nabla\phi_2=\gamma \quad on\ S$$

Let's now define $\phi_0=\phi_1-\phi_2$

$$\left.\begin{aligned}\nabla^2\phi_0&=\nabla^2(\phi_1-\phi_2)=\overbrace{\nabla^2\phi_1}^{\rho}-\overbrace{\nabla^2\phi_2}^{\rho}=0\\ \hat{n}\cdot\nabla\phi_0&=\hat{n}\cdot(\underbrace{\nabla\phi_1}_{\gamma}-\underbrace{\nabla\phi_2}_{\gamma})=0 \quad on\ S\end{aligned}\right\}\Rightarrow \hat{n}\cdot\nabla\phi_0=0\Rightarrow\phi_0\nabla\phi_0\cdot\hat{n}=0 \quad on\ S\Rightarrow\iint_S\phi_0\nabla\phi_0\cdot\hat{n}dS=0$$

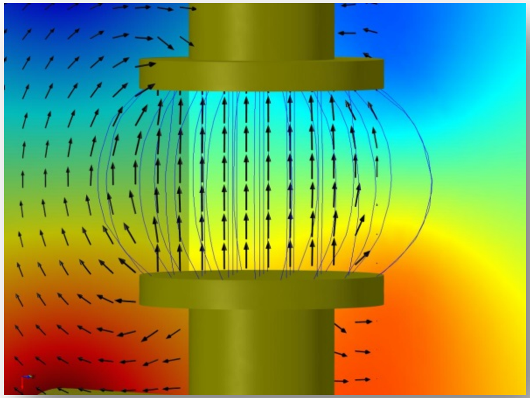
$$0=\iint_S\phi_0\nabla\phi_0\cdot\hat{n}dS=\iiint_V\nabla\cdot\phi_0\nabla\phi_0 dV=\iiint_V\underbrace{(\nabla\phi_0)^2}_{\geq 0}dV \Rightarrow \nabla\phi_0=0 \Rightarrow \phi_0=const.$$
$$\Rightarrow \phi_1=\phi_2+const.$$

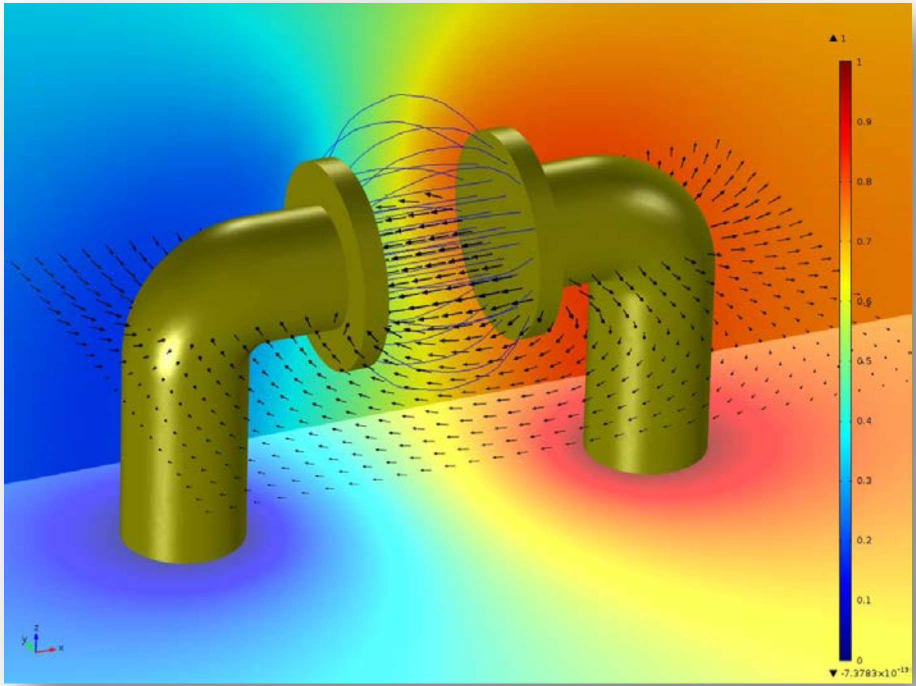
Gauss' theorem

see proof of theorem 1

THE CAPACITOR EXAMPLE

Thanks to Pablo (and COMSOL)





Color plot: Potential V, Arrows: Electric field, Streamlines: Electric field, Gold: Grounded and positive electrode

Laplace equation

$$\nabla^2 V = 0$$

Boundary conditions:

- Left electrode

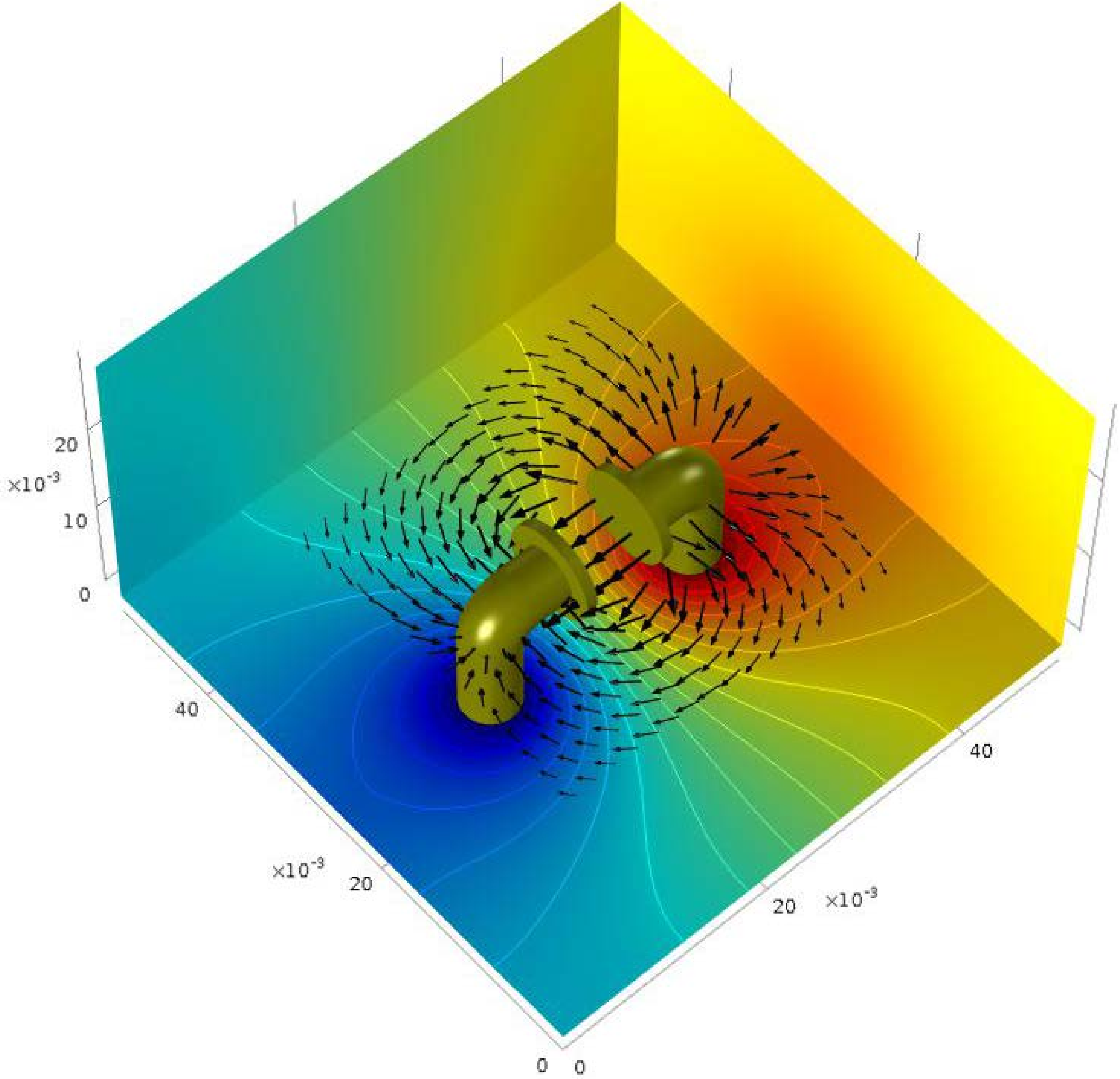
$$V = 0 \qquad \textit{(Dirichlet BC)}$$

- Right electrode

$$V = 1 \qquad \textit{(Dirichlet BC)}$$

- To solve the problem, COMSOL needs boundary conditions on the floow. For this example, insulating boundary condition on the floor have been applied: (Neumann)

$$\overline{n} \cdot \nabla V = 0 \qquad \textit{(Neumann BC)}$$



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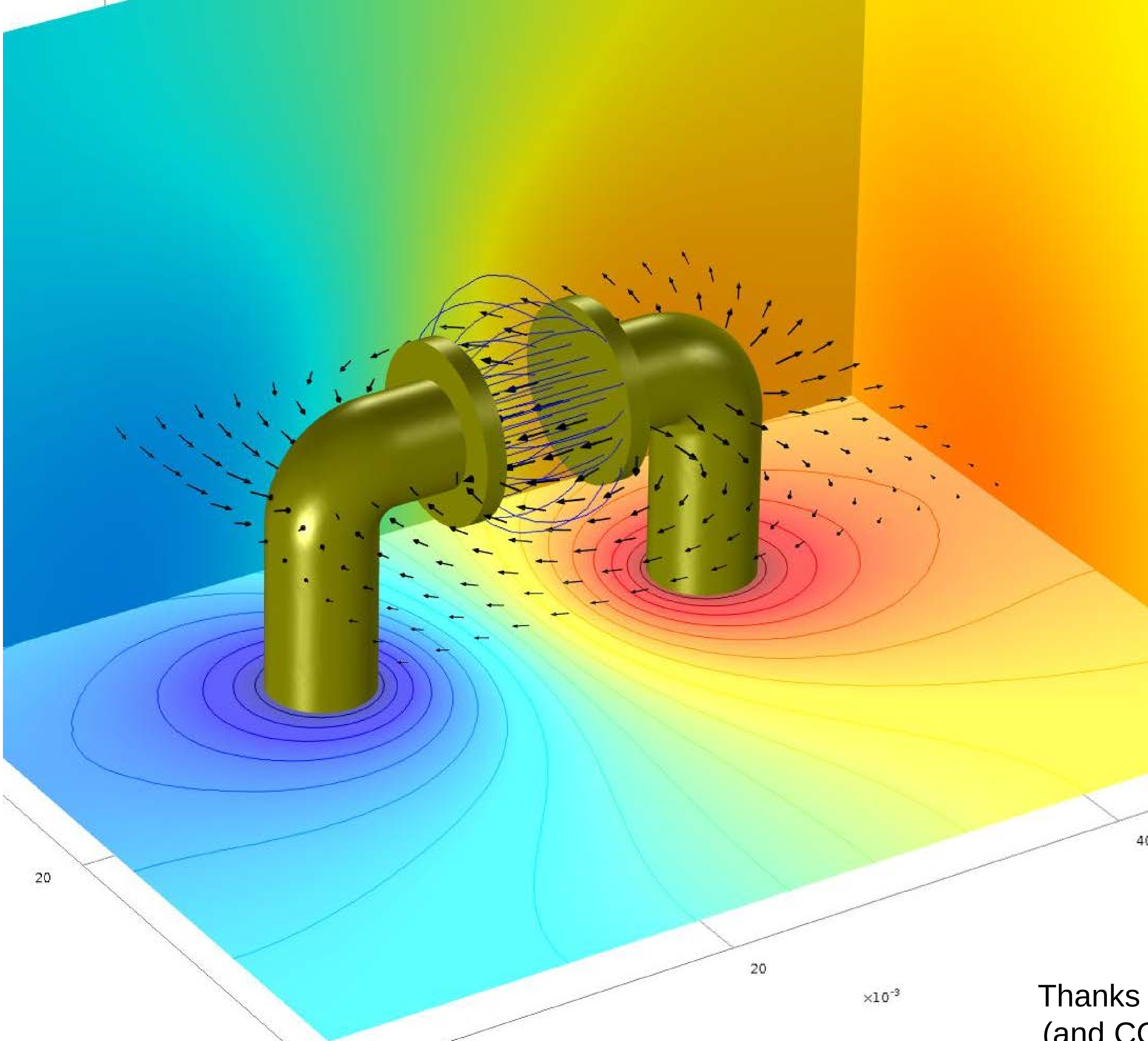

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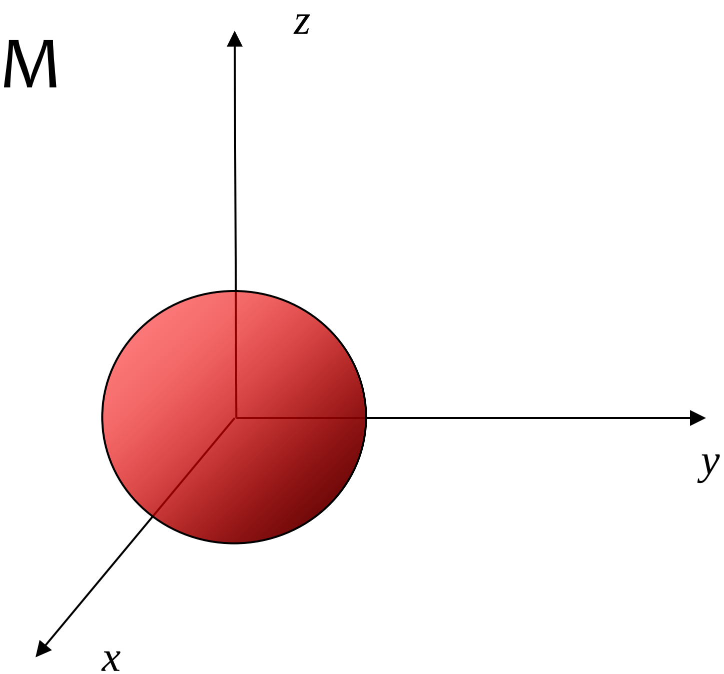
TARGET PROBLEM

A sphere has radius R and charge density $\rho=\rho_c$.
Calculate:
- the electric field and
- the electrostatic potential

inside and outside the sphere.



Spherical symmetry: $\phi = \phi(r)$

Outside the sphere

$$\nabla^2\phi_E = 0 \quad \Rightarrow \quad \phi_E^{out}(r) = -\frac{a}{r} + b$$

typically

$$\lim_{r\to\infty}\phi_E(r) = 0 \quad \Rightarrow \quad b = 0$$

$$\bar{E} = -\nabla\phi_E = -\left(\frac{d\phi_E(r)}{dr}, \frac{1}{r}\frac{d\phi_E(r)}{d\theta}, \frac{1}{r\sin\theta}\frac{d\phi_E(r)}{d\varphi}\right) \quad \Rightarrow E_r^{out} = -\frac{d\phi_E^{out}(r)}{dr} = -\frac{a}{r^2}$$

Inside the sphere

$$\nabla^2\phi_E = -\frac{\rho_c}{\varepsilon_0} \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_E(r)}{dr}\right) = -\frac{\rho_c}{\varepsilon_0}$$

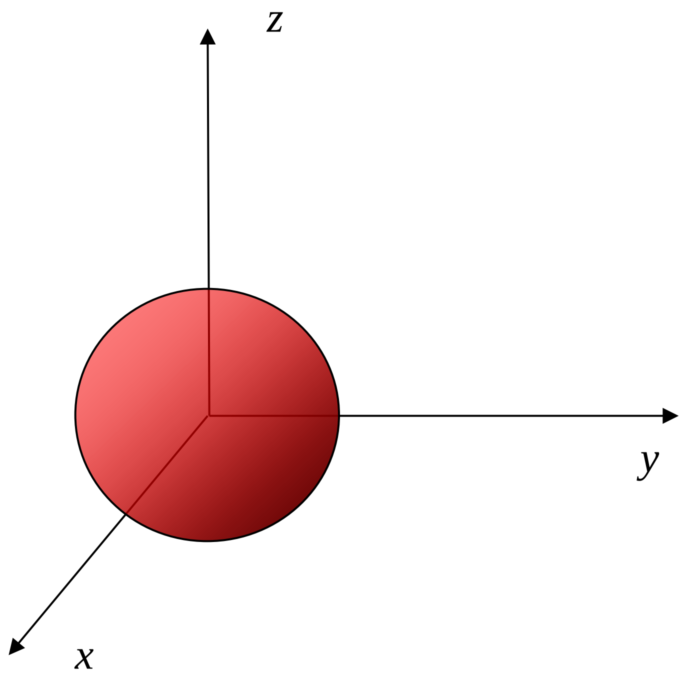
multiplying by r^2 and integrating:

$$r^2\frac{d\phi_E(r)}{dr} = -\frac{\rho_c r^3}{3\varepsilon_0} + c \;\Rightarrow\; \frac{d\phi_E(r)}{dr} = -\frac{\rho_c r}{3\varepsilon_0} + \frac{c}{r^2} \quad \Rightarrow \phi_E^{in}(r) = -\frac{\rho_c r^2}{6\varepsilon_0} + d$$

$$E_r^{in} = -\frac{d\phi_E^{in}(r)}{dr} = +\frac{\rho_c r}{3\varepsilon_0} - \frac{c}{r^2}$$

Divergent at r=0
NOT physical! $\Rightarrow$ c=0

TARGET PROBLEM



We still have to calculate a and d!

Boundary conditions:

$$E_r^{out}\left(R\right)=E_r^{in}\left(R\right) \quad \Rightarrow \quad -\frac{a}{R^2}=\frac{\rho_c R}{3\varepsilon_0} \quad \Rightarrow \quad a=-\frac{\rho_c R^3}{3\varepsilon_0}$$

$$\phi_E^{out}\left(R\right)=\phi_E^{in}\left(R\right) \quad \Rightarrow -\frac{\rho_c R^2}{6\varepsilon_0}+d=\frac{\rho_c R^3}{3\varepsilon_0 R} \quad \Rightarrow \quad d=\frac{\rho_c R^2}{2\varepsilon_0}$$

$$\boxed{\phi_E^{out}\left(r\right)=\frac{\rho_c R^3}{3\varepsilon_0 r}} \qquad \boxed{E_r^{out}=+\frac{\rho_c R^3}{3\varepsilon_0 r^2}}$$

$$\boxed{\phi_E^{in}\left(r\right)=\frac{\rho_c R^2}{6\varepsilon_0}\left(3-\frac{r^2}{R^2}\right)} \qquad \boxed{E_r^{in}=+\frac{\rho_c r}{3\varepsilon_0}}$$